Strain analysis and structural optimization of functionally graded rod with small concentration of inclusions (MAT303-15)

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Abstract. The homogenization procedure is applied to strain analysis and optimal design of a Functionally Graded (FG) rod in the case when the inclusion size is essentially less than a distance between them. The method is illustrated using an example of the rod longitudinal strain. We considered separately the cases of FG inclusion sizes and FG steps between inclusions. Two particular problems of optimal design are discussed in some details.

1. Introduction

A fundamental approach allows deducing the macro-scale laws and the constitutive relation by properly homogenization over the micro-scale is known as the homogenization method [1-9]. This method also used to modeling and simulating mechanical behavior of FG Materials (FGM) and FG Structures (FGS) [10-15]. FGMs are composites consisting of two different materials with a gradient composition. For homogenization method coefficients of regular composites state equations are usually [1-4] approximated by the first terms of their Fourier series (Fig. 1).

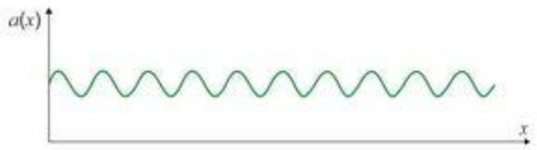

Figure 1. Example of equation coefficient for regular composites.

Similarly [10-15] can be approximated the coefficients of FGSs state equations with FG inclusion sizes **(**Fig. 2) and FG step between the inclusions (Fig. 3).

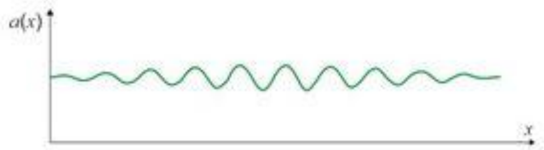

Figure 2. Example of equations coefficient for composite with FG inclusion sizes.

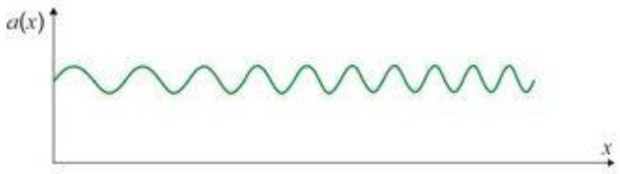

Figure 3. Example of equation coefficient for composite with FG steps between inclusions.

However, the truncated Fourier parts even for a small number of terms relatively good approximate the coefficients of the state equations for large concentration of inclusions (fibers, cells, etc.), when the distance between inclusions have been the same order as their typical size. For small concentration, when the distance between inclusions is essentially larger than their size, the state equations coefficients are approximated by impulsive periodic function (see, for instance, Fig. 4). In this case usual homogenization procedure may meet some problems to be directly applied.

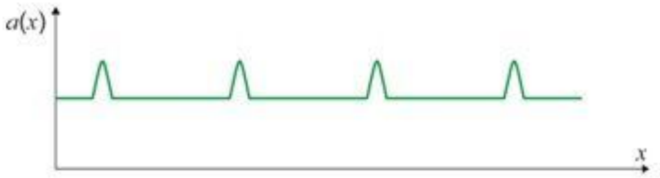

Figure 4. Example of coefficient for composites with small inclusion concentration.

Therefore, for a small concentration of inclusions it is recommended to use the further presented variant of homogenization method, where used small size of the inclusions with respect to the distance between them for asymptotic procedure. Modifications of this approach for FGS with small inclusion concentrations are proposed.

The applied method is illustrated using a relatively simple problem, i.e. a rod with a longitudinal strain. Rod diameter is taken commensurable with inclusions dimension.

2. FG inclusion sizes

FG properties can be achieved, for instance, with respect to different inclusion sizes. Let us analyse an influence of different sizes of inclusions on the longitudinal rod stiffness, keeping constant the distance between inclusions (Fig. 5). We define changes of the inclusion dimensions by a function $V = V(x)$.

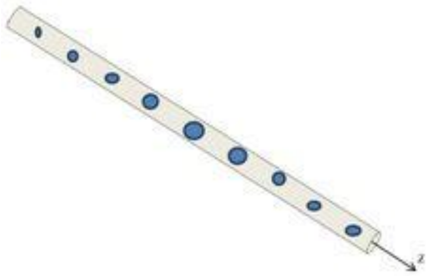

Figure 5. Rod with FG inclusions.

We approximate inclusions (Fig. 5) by concentrated elastic elements (Fig. 6), where stiffness $k_1(z)$ characterizes the inclusions influence. Observe that for composites with regular structure in the analogous models of two-component rod are applied (see [16-18]).

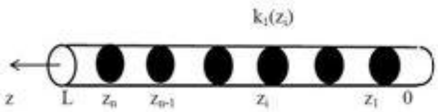

Figure 6. Two-component rod with concentrated elastic elements.

Obviously, a number of inclusions n is large, and hence the distance $l = z_i - z_{i-1}$ between them is much less than the rod length L, l<<L. Therefore, in order to investigate the longitudinal deformation of the two components rod (Fig. 6), one may apply the following variant of the homogenization procedure.

Equilibrium equation of the rod part between the inclusions has the following form

$$\frac{d^2u}{dx^2} = q, \tag{1}$$

where $x = z/l$; $u = v/l$; v is the longitudinal displacement; $q = \frac{p(x)}{lk_0}$; $p(x)$ is the applied load; $k_0=E_0F$; E_0 is the Young modulus matrix; F is the cross-sections area.

Compatibility conditions regarding the i-th elastic cross-sections are as follow

$$(u)^- = (u)^+; \quad \left(\frac{du}{dx}\right)^- - \left(\frac{du}{dx}\right)^+ = ku, \tag{2}$$

where $(\dots)^- = \lim_{x=i-0}(\dots)$; $(\dots)^+ = \lim_{x=i+0}(\dots)$; $k = \frac{lk_1(x)}{k_0}$.

3. Homogenization procedure

Owing to the multiscale homogenization approach, let us introduce the fast variable ξ

$$\xi = x / \varepsilon, \tag{3}$$

where ε=1/n<<1.

We treat the variables x and ξ as independent ones, and the differential operator occurred in (1), (2) has the following form

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{4}$$

Displacement u can be presented in the following form

$$u = u_0(x) + \varepsilon^2 u_1(x,\xi) + \varepsilon^3 u_2(x,\xi) + ..., \tag{5}$$

where u_s (s=1,2,...) is periodic with respect to ξ.

Substituting formulas (4), (5) into equation (1) and condition (2), and carrying out the splitting with respect to ε, the following equations are obtained

$$\frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial x^2} = q, \tag{6}$$

$$(u_1)_{\xi=0} = (u_1)_{\xi=n}, \tag{7}$$

$$\left(\frac{\partial u_1}{\partial \xi}\right)_{\xi=0} - \left(\frac{\partial u_1}{\partial \xi}\right)_{\xi=n} = \varepsilon^{-1} k u_0. \tag{8}$$

We assumed $\varepsilon^{-1}k \sim 1$ while deriving formula (8).

Integrating (6) with regard to ξ and defining the integration constant via condition (7), we get

$$\frac{\partial u_1}{\partial \xi} = \left(q - \frac{d^2 u_0}{dx^2}\right)\left(\xi - \frac{n}{2}\right). \tag{9}$$

Substituting formula (9) into condition (8), the following homogenized equation describing the longitudinal displacement of the two-component rod is obtained

$$\frac{d^2 u_0}{dx^2} + k(x) u_0 = q. \tag{10}$$

Micromechanical effects are described by functions u_s (s=1,2,...).

4. Inverse problem

The main advantage of the proposed approach is that it allows efficiently solving the problems of optimization, i.e. problems devoted to determination of optimal characteristics of the internal material structure protecting the given structure properties. In the studied case of the FG amplitudes, the target characteristic is the function $V=V(x)$ governing a rule of the inclusion dimensions change.

As an example we consider the problem of determination of the function $V(x)$ protecting the larger longitudinal stiffness for a given load $q(x)$. It is convenient to take rather the function $k(x)$ as the control function instead of the function $V(x)$.

Without lose of generality let us take the boundary conditions in the following form

$$u_0(0)=0, \quad \left.\frac{du_0}{dx}\right|_{x=n}=0. \tag{11}$$

In order to measure the rod stiffness properties we take energy of the elastic deformations and use zero order approximation of the displacement (10). Then, we define a minimum of the following functional

$$I=\int_0^n q u_0 dx \to \min_k . \tag{12}$$

One can introduce the following isoperimetric condition which guarantees a constant total inclusion volume

$$\sum_{i=1}^{n} k(x_i) = C - const. \tag{13}$$

Condition (13) can be transformed to the isoperimetric form owing to application of an Euler-Maclaurin formula [19]

$$\sum_{i=1}^{n} k(x_i)=\int_0^L k(x)dx+\frac{1}{2}(k(0)+k(L))-\frac{1}{12}(k'(0)+k'(L))+\frac{1}{720}(k''(0)+k''(L))-\dots .$$

For large number of inclusions $n \gg 1$ and smooth function $k'(x)\sim 1,$ we can neglect non-integral terms

$$\int_0^n k(x)dx=c. \tag{14}$$

In practice, the inclusion sizes meet the technological constrains. Hence, next constraint for the target function is required

$$k_{\min} \le k(x) \le k_{\max}. \tag{15}$$

Constraint (15) is satisfied through introduction of the following new control function θ (x) [20]:

$$k = \alpha + \gamma \sin\theta, \tag{16}$$

where: $\alpha = 0.5(k_{\min} + k_{\max}),\ \ \gamma = 0.5(k_{\min} + k_{\max}).$

Obtaining of the function $\theta(x)$ requires solution to the following inverse problem (10)-(16):

$$I = \int_0^n q u_0 dx \to \min_{\theta}, \tag{17}$$

$$I_1 = \gamma \int_0^n \sin\theta dx = c, \tag{18}$$

$$\frac{d^2 u_0}{dx^2} + (\alpha + \gamma \sin\theta) u_0 = q, \tag{19}$$

$$u_0(0) = 0, \qquad \left(\frac{du_0}{dx}\right)_{x=n} = 0. \tag{20}$$

Defining first variations of integrals (17), (18), and a variation equation corresponding to (19) with the boundary conditions (20), one obtains

$$\delta I = \int_0^n q \delta u_0 dx, \qquad \delta I_1 = \gamma \int_0^n \cos\theta \delta\theta dx, \tag{21}$$

$$\frac{d^2 \delta u_0}{dx^2} + (\alpha + \gamma \sin\theta)\delta u_0 + \gamma \cos\theta u_0 \delta\theta = 0, \tag{22}$$

$$\delta u_0(0) = 0, \qquad \left(\frac{d\delta u_0}{dx}\right)_{x=n} = 0. \tag{23}$$

The variation of Lagrange functional follows

$$\delta I + \lambda \delta I_1 + \int_0^n \tilde{v} \left(\frac{d^2 \delta u_0}{dx^2} + (\alpha + \gamma \sin\theta)\delta u_0 + \gamma \cos\theta u_0 \delta\theta \right) dx = 0, \tag{24}$$

where λ denotes the Lagrange multiplier, and $\tilde{v}$ is the conjugated variable.

Conjugated variable $\tilde{v}$ being defined through the condition minimizing functional (17) should not contain the variation δw_0 . For this purpose, we integrate by parts the first term of the integrand two times taking into account conditions (23). Non-integral terms are equal zero if the following boundary conditions are applied to the conjugated variable

$$\tilde{v}(0)=0, \qquad \left(\frac{d\tilde{v}}{dx}\right)_{x=n}=0. \tag{25}$$

Finally, equation (24) takes the following form

$$\int_0^n\left(\left(\frac{d^2\tilde{v}}{dx^2}+(\alpha+\gamma\sin\theta)v+q\right)\delta u_0+\gamma\cos\theta(\tilde{v}u_0+\lambda)\delta\theta\right)dx=0. \tag{26}$$

In order to keep variation δJ independent on δu_0, the following equation should be satisfied

$$\frac{d^2\tilde{v}}{dx^2}+(\alpha+\gamma\sin\theta)\tilde{v}+q=0, \tag{27}$$

and the following optimality condition takes place

$$\cos\theta(\tilde{v}u_0+\lambda)=0. \tag{28}$$

Comparison of the boundary condition (22), (23) and (25), (27) yields

$$\tilde{v}=-u_0. \tag{29}$$

Taking into account equation (29), optimality condition is result to the following form

$$\cos\theta(u_0^2-\lambda)=0. \tag{30}$$

As it has been pointed out in references [10-15], the occurrence of singular points belongs to typical problems, while solving the problems of optimization of FGS. If one is looking for the control function $\theta(x)$ as continuous one, then it is impossible to satisfy the boundary condition (20) for $x=0$. Therefore, we assume the control function $\theta(x)$ in the form of piece-wise continuous function, which in the $(0,x_1)$ satisfies the following condition

$$\cos\theta=0, \tag{31}$$

and on (x_1,L) we have

$$u_0^2 - \lambda = 0. \tag{32}$$

One gets a co-ordinate of the point x_1 from continuity condition in this point of both the function u_0 and its derivative u_{0x} (these conditions are yielded by the external Weierstrass-Erdmann relations [21]):

$$\lim_{x\to x_1-0} u_0 = \lim_{x\to x_1+0} u_0, \qquad \lim_{x\to x_1-0} \frac{du_0}{dx} = \lim_{x\to x_1+0} \frac{du_0}{dx}. \tag{33}$$

The conditions (33), taking into account relations (31), take the following form

$$u_{01}(x_1) = \pm\sqrt{\lambda}, \tag{34}$$

$$\left(\frac{du_{01}}{dx}\right)_{x=x_1} = 0, \tag{35}$$

where u_{01} denotes displacement in the interval $(0, x_1)$. Relations (16), (19), (31) allow to derive the following equation

$$\frac{d^2 u_{01}}{dx^2} + k_{\min} u_{01} = q, \tag{36}$$

with BCs (11), (35). Assuming that u_{01} is known, conditions (34) allow describing x_1 via λ. Substituting conditions (32) into equation (19), we find a formula for $\sin\theta$ in the interval (x_1, L). Then, taking into account formula (18), the following optimal control function is defined

$$k = \begin{cases} k_{\min}, & x \in (0, x_1), \\ \pm\dfrac{q}{\sqrt{\lambda}}, & x \in (x_1, L). \end{cases} \tag{37}$$

The Lagrange constant λ is found from an isoperimetric condition (14), which takes the following form

$$\int_{x_1}^{n} q\,dx = \pm\sqrt{\lambda}\,(c - k_{\min} x_1). \tag{38}$$

5. Example of amplitude optimization

In order to illustrate the proposed method, we consider the problems (10)-(15) for linearly distributed load, i.e.

$$q = \rho x, \qquad \rho = const. \tag{39}$$

Assume that the minimal size of inclusions is equal to zero, i.e.

$$k_{\min} = 0. \tag{40}$$

Formulas (39), (40), taking into account the boundary conditions (36), (11), (35), yield

$$u_{01} = \rho\left(\frac{x^3}{6} - \frac{x_1^2}{2}x\right). \tag{41}$$

Equation (34) gives

$$\sqrt{\lambda} = \frac{\rho}{3}x_1^3. \tag{42}$$

Formula (38) taking into account (39), (40), (42), implies the following equation to find x_1:

$$2cx_1^3 + 3x_1^3 - 3L^2 = 0. \tag{43}$$

Observe, that equation (43) does not include an intensity of the load ρ, i.e., the variable x_1 depends only on the applied load character, which is typical for the linear statement formulation. We take the following parameters in order to carry out the numerical computations

$$n = 100, \qquad c = 10^2 \cdots 10^3. \tag{44}$$

It should be emphasized that for the chosen parameters equation (43) uniquely defines x_1, since among any three roots of the equation, only one is positive.

The function x_1 versus c is reported in Figure 7.

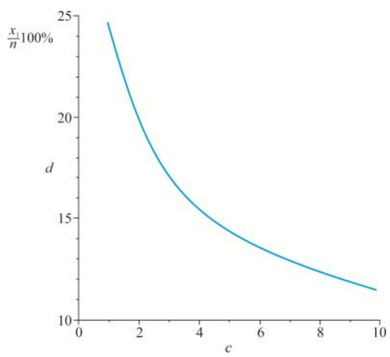

Figure 7. The function x_1 versus c (x_1 denotes a boundary area without inclusions).

Finally, (37) yields the optimal control function

$$k = \begin{cases} 0, & x \in (0, x_1), \\ \dfrac{3\rho x}{x_1^3}, & x \in (x_1, n). \end{cases} \tag{45}$$

Let us estimate the efficiency of the proposed optimization. For this purpose we compare the extension of the rod for the optimal stiffness distribution for the equivalent cross-section (45) and extension of the rod of the regular form. Extension Δ_0 of the optimal rod can be found from equations (32), (42)

$$\Delta_0 = \frac{\rho x_1^3}{3}. \tag{46}$$

Extensions of the regular form rod Δ can be found from the boundary value problem (10), (11), (39) for $\bar{k} = c/n$

$$\Delta = \frac{\rho n^3}{c}\left(1 - \frac{\tan\sqrt{c}}{\sqrt{c}}\right). \tag{47}$$

With a help of relations (46), (47) we find the relative decrease of the rod extension due to optimal distribution of the inclusions volume

$$\delta = \frac{\Delta - \Delta_0}{\Delta} 100\%. \tag{48}$$

For the considered parameters (44) the relative decrease of the extension δ is of amount 50%, i.e. for $c = 10^2$, $\delta = 46.6\%$; $c = 5 \cdot 10^2$, $\delta = 49.1\%$; $c = 10^3$, $\delta = 49.7\%$.

6. FG steps between inclusions

Now we are aimed to analyse an influence of FG steps between equal inclusions. Consider the basic two-component rod (Fig. 6) with equal elastic inclusions, $k = const$. Assuming a fixed number of inclusions n and $L = n$, we have $l = 1$. Now we change the distance l. In order to describe the rule of the step changes, and following reference [15], we apply a steps function $f(x)$, such that

$$f(x_i) = i, \tag{49}$$

where x_i is a co-ordinate of the *i*-th inclusion.

Properties of the steps function protecting the constant number of inclusions can be defined via Fig. 8, which can be treated as a nomogram to define co-ordinates of the inclusions versus function steps.

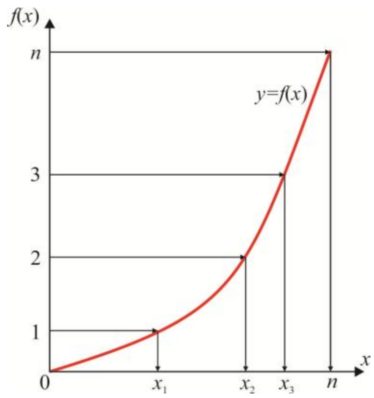

Figure 8. Nomogram to define co-ordinates of the *i*-th elastic inclusions x_i

for a given steps function $f(x)$.

Figure 8 implies that in order to protect constant number of inclusions, the function $f(x)$ should have the following properties

$$f(0) = 0, \quad f(n) = n, \quad f'(x) \geq 0. \tag{50}$$

Relation $\Delta f \approx f'(x)\Delta x$ yields approximation to the steps between inclusions. For the given step of the basic rod (Fig. 6) $\Delta f = 1$, the non-constant step of the FG rod $\Delta x = s$ takes the following form

$$s \approx \frac{1}{f'(x)}. \tag{51}$$

7. Direct problem for FG steps

Proceeding in the similar way to that used for the FG inclusion sizes, we now consider the inclusions thickness approaching to zero, and the equations governing the rod deformation with the FG steps takes the following form

$$\frac{d^2u}{dx^2} + k\sum_{i=1}^{n}\delta(f(x_1) - i)u = q, \tag{52}$$

where $\delta(x)$ denotes the Dirac delta function.

We introduce the following variable $\eta = f(x)$ $(x = f^{-1}(\eta))$. Therefore, equation (52) is recast to the following form

$$\frac{d}{d\eta}\left(\frac{du}{\varphi d\eta}\right) + k\sum_{i=1}^{n}\delta(\eta - i)\varphi u = Q, \tag{53}$$

where: $\varphi = \frac{d(f^{-1})}{d\eta}$, $Q = \varphi q$.

Equation (53) presents an equation with periodically discontinuous coefficients, and we may apply a homogenization procedure. Equation of equilibrium between inclusions (1) and compatibility conditions (2) for the considered FG steps take the following form

$$\frac{d}{d\eta}\left(\frac{du}{\varphi d\eta}\right) = Q, \tag{54}$$

$$(u)_{\eta=i-0} = (u)_{\eta=i+0}, \qquad \left(\frac{du}{\varphi d\eta}\right)_{\eta=i-0} - \left(\frac{du}{\varphi d\eta}\right)_{\eta=i+0} = ku. \tag{55}$$

After introduction of the fast variable $\xi = \eta / \varepsilon$ and applying asymptotic series development regarding displacements

$$u = u_0(\eta) + \varepsilon^2 u_1(\eta,\xi) + \varepsilon^3 u_2(\eta,\xi) + ..., \tag{56}$$

relations (54), (55) yield the following homogenization equations for u_0 :

$$\frac{d}{d\eta}\left(\frac{du_0}{\varphi d\eta}\right) + ku_0 = Q \tag{57}$$

and the equation for the correction term u_1 :

$$\frac{\partial u_1}{\partial \xi} = \varphi\left(Q - \frac{d}{d\eta}\left(\frac{du_0}{\varphi d\eta}\right)\right)\left(\xi - \frac{n}{2}\right). \tag{58}$$

If in the homogenization equation (57) we return to the original variable, we get

$$\frac{d^2 u_0}{dx^2} + kf'(x)u_0 = q. \tag{59}$$

It should be emphasized that the obtained homogenized equation (59) represents all physical aspect of the problem. The non-differentiable term occurring in the left hand side of this equation presents an "additional stiffness" governed by inclusions and continuously distributed along the rod length. In the case of FGS, this distribution is not uniform. In the case of FG steps, more dense localization of inclusions involves larger contribution of the "additional stiffness". Mathematically, this property is described by the derivative of the steps function $f'(x)$ in equation (59).

8. Inverse problem for FG steps

Consider the inverse problem for equation (59) with the boundary conditions (11), take the following control function

$$\psi = kf'(x). \tag{60}$$

Minimizing functional *I* represents the energy of elastic deformation

$$I = \int_0^n u_0 q dx \to \min_{\psi}. \tag{61}$$

Condition of keeping constant the number of inclusions (50) yields the isoperimetric form for the control function

$$\int_0^n \psi dx = kn. \tag{62}$$

We apply also the technological bounds for ψ, analogous to (15), which will be satisfied automatically after introduction of the control function (16). One may easily convince that the considered inverse problem for the FG rod (59)-(62) coincides with the analogous problem (10)-(16). In what follows we illustrate how to find solution to the considered problem.

9. Example of steps optimization

We consider optimization of the steps for the rod (39). Similarly to condition (40), the following formula holds

$$\psi_{\min} = 0. \tag{63}$$

It means, that inclusions do not appear in the interval $(0, x_1)$. The illustrated requirement (63) yields the particular case considered in section 5 (for $k = \psi$). After defining the control function due to formula (45), the function $f(x)$ defining the optimal coordinated of the inclusion constant is estimated with a help of the second condition of (50). We get finally

$$f(x)=\begin{cases}0, & x\in(0,x_1),\\ \dfrac{3\rho(x^2-n^2)}{2x_1^3k}+n, & x\in(x_1,n).\end{cases} \tag{64}$$

The function $f(x)$ for $\rho=100$, $k=10$, $n=100$ is shown in Figure 9.

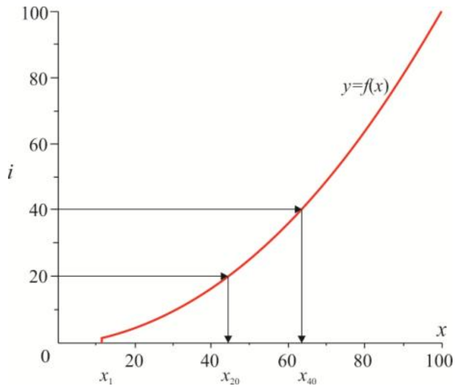

Figure 9. Nomogram for determination of the optimal coordinates of inclusions protecting the larger longitudinal rod stiffness for the linearly distributed load.

Substituting equation (49) into (64) yield optimal coordinates of the inclusions:

$$x_i=\sqrt{n^2-\frac{2x_1^3k(n-i)}{3\rho}}, \tag{65}$$

where: $i=1...100$, and x_1 is defined by equation (43).

In Figure 10 the a window enlargement of the nomogram part (Fig. 9) is shown for two first inclusions.

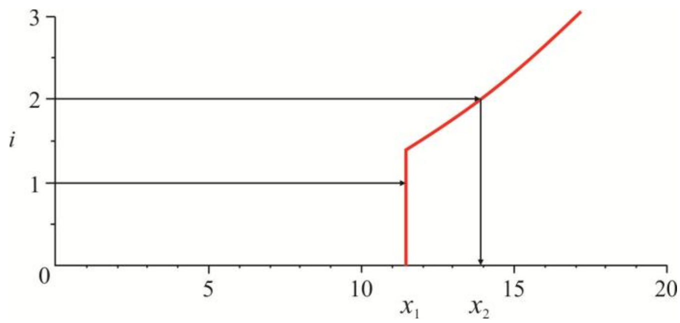

Figure 10. Scheme for determination of the coordinate of two first inclusions x_1 and x_2.

The first inclusion coordinate ($x_1 = 11.45$) is found from equation (43), whereas the second inclusions coordinate $x_2 = 13.88$ is given by formula (65).

Concluding remarks

Introduction the step function (49) allowed solving problems of computations and optimal design of FGS with a FG inclusion dimensions and FG steps between inclusions using the unique approach. Both considered problems occurred identical from the mathematical points of view, whereas the difference between them consists in the meaning of coefficients in the state equations and control functions.

While optimization the FGS with FG inclusion sizes and FG steps between inclusions it is recommended to search the control function on a set of piece-wise continuous functions. The optimization process is realized with a help of two mechanisms. First, we define boundary area where inclusions do not occur. Second, in the case of FG inclusion sizes we interlace inclusion dimensions to fit a rule of the external load distribution. The reported optimization mechanisms, being obvious from physical point of view, have found a mathematical foundation in our work. It is expected that the proposed method will be also sufficient during computations and optimal design of more complex heterogeneous structures governed by the differential equations of higher order.

A second direction of the proposed method development is the FGS, where FG inclusion sizes and FG steps between inclusions appear simultaneously.

Acknowledgments

This work has been supported by the Polish National Science Centre, MAESTRO 2, No. 2012/04/A/ST8/00738.

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